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Use of Finite Difference Approximations to Partial Differential Equations for Problems Having Boundaries at Infinity

***Abstract*—A computationally simple technique is presented for solving finite difference equations arising from potential problems, part of whose boundary is at infinity. The procedure makes use of an arbitrary "fictitious" boundary drawn close to the regions of physical interest. An initial guess is made of the potential on this boundary as well as at all interior points. Well-known iterative techniques are used to correct the values of the interior potentials. Meanwhile the potentials on the boundary are corrected iteratively by recalculating them from the sources or charges in the entire region, which in turn are calculated from the current iteration of the interior potential.**

The technique is valid even if parts of the physical structure, such as an air–dielectric interface in microstrip, extend toward infinity. The fictitious boundary need not include all of the structure, providing the rate of falloff of the sources outside the boundary is known.

Introduction

A recent issue of this TRANSACTIONS [1] is devoted to computer oriented microwave practices. Several papers deal with the technique of using successive overrelaxation (SOR) of finite difference approximations to solve partial differential equations. This correspondence describes a modification of the SOR technique for treating problems, part of whose boundary is at infinity, without the necessity of using an inordinately large grid of points.

Review of SOR Technique

In the SOR method the region within the specified boundary is divided up into a grid (usually rectangular) of points. The differential operator is replaced by a finite difference operator and the differential equation is replaced by a difference equation at each point of the grid. For an $N \times N$ grid, the partial differential equation is replaced by N^2 simultaneous linear equations. If, for example, the original equation is Laplace's equation

$$\nabla^2 \phi = 0 \tag{1}$$

the finite difference equation at each point is

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0 = 0 \tag{2}$$

or

$$\phi_0 = (\phi_1 + \phi_2 + \phi_3 + \phi_4)/4 \tag{3}$$

where the geometrical relationship of the points is shown in Fig. 1. This set of simultaneous equations is solved by an iterative process. Each step of this process involves evaluating a new potential at each point of the grid from the finite difference equation and using this new value in the calculation of all subsequent potentials. The process ends when no further changes in potentials occur on successive iterations, i.e., when the finite difference equations are satisfied at every point. In practice, a larger correction is made than is called for by the difference equation in order to speed convergence; hence the name successive overrelaxation.

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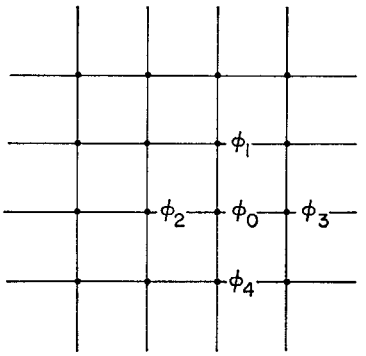

Fig. 1. Arrangement of points used for ∇^2 operator.

Thus (3) is replaced by

$$\phi_{0\,\text{new}} = \{(\phi_1 + \phi_2 + \phi_3 + \phi_4)/4 - \phi_{0\,\text{old}}\}\omega + \phi_{0\,\text{old}} \tag{4}$$

for each step of the iteration. The relaxation parameter ω is generally taken between 1 and 2. The boundary condition is taken into account by including it in the difference equation for the points of the grid either on or immediately adjacent to the boundary. The exact procedure depends on the nature of the boundary condition.

Open Boundaries

In many microwave configurations that are currently under active consideration (such as microstrip), the region in which the potential is nonzero extends to infinity. The problem can be solved by finite differences by imposing a boundary condition of $\phi = 0$ at some finite distance from the important parts of the structure. However, if accurate results are to be obtained, this boundary must be quite far out, and hence the grid will have a very large number of points and the solution will waste excessive computer time. An attempt to avoid this problem, using an artificial boundary much closer to the region of interest, has been proposed by Cermak and Silvester [2]–[4] and elaborated on by Richter [5]. Using a rather involved calculation, they obtain the potential on this boundary and hence are able to find the potential inside it by ordinary SOR techniques. We present here a procedure for finding this potential which we consider conceptually simpler and easier to handle in a practical problem. Moreover, our technique also permits a simple and accurate approximate solution for cases in which some sources are outside the boundary. Thus very wide (or infinitely wide) geometries can often be treated with a relatively close boundary.

The artificial boundary is a fictitious boundary drawn outside the region of practical interest. Given the true potential on this boundary, we can solve for the potentials inside it by established techniques, as discussed above. On the other hand, given the potentials inside the boundary, we can calculate the charges on all conductors and dielectric interfaces within the boundary and hence can calculate also the potential at the boundary due to these charges. When the true potential at the boundary is used to calculate the charges, then the potential calculated from the charges will reproduce the true potential. But if an incorrect potential had been assumed for the boundary, then the boundary potential recalculated from the charges would not be the same, because we do not include contributions to the potential due to charges on the fictitious boundary. For the true solution, of course, there are no charges on the fictitious boundary, since it is just a mathematical construction. We can thus generate a self-consistent iterative technique to find both the boundary potential and the charges. Specifically, we make an initial guess of the potential on the boundary and use this to calculate a first guess of the potential in the interior. We then use this interior potential to calculate the charges in the interior and a new potential on the boundary. We continue this process, using each successive iteration of the interior potential to make the next calculation of the boundary potential and vice versa.

To show how this procedure can succeed and that it is not in conflict with the uniqueness theorem, let us consider a simple electrostatic problem with a well-known answer, namely, a sphere of radius

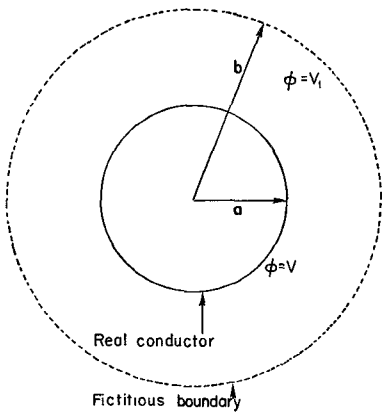

Fig. 2. Geometry of sample electrostatic problem. Real conductor is a sphere of radius a at potential V with respect to infinity. Fictitious boundary is a larger concentric sphere of radius b with potential values V_i, $i = 0, 1, \cdots$, for successive iterations.

a at potential V. The correct potential is, of course,

$$\phi = (a/r) \cdot V, \qquad r \geq a. \tag{5}$$

For our fictitious boundary let us choose a larger concentric sphere of radius b as shown in Fig. 2. Following the procedure described above, we make a guess for the potential at the fictitious boundary. For simplicity let us take it to have a constant value V_0 (a computer would, naturally, assume a numerical value, but we may assume an algebraic value). From elementary electrostatic theory, the potential is

$$\phi = A_0/r + B_0, \qquad a \leq r \leq b \tag{6}$$

with

$$A_0 = b\xi(V - V_0) \tag{7}$$

and

$$B_0 = b\xi(a^{-1}V_0 - b^{-1}V) \tag{8}$$

where ξ is defined as

$$\xi = a/(b - a). \tag{9}$$

Using this potential we calculate the charge on the real sphere using Gauss' law:

$$Q = \frac{1}{4\pi}\int_S \left(-\frac{\partial\phi}{\partial n}\right)_{r=a} dS \tag{10}$$

$$= A_0 = b\xi(V - V_0). \tag{11}$$

The potential due to this charge is

$$\phi = A_0/r, \qquad r \geq a \tag{12}$$

and at the fictitious boundary has the value

$$V_1 = A_0/b = \xi(V - V_0) \tag{13}$$

which we use for our next iteration of the surface potential. Using this new boundary potential, we get a new interior potential of the same form as (6), but with new values of A and B. The new value for A is

$$A_1 = b\xi(V - V_1) = b\xi\{V - \xi(V - V_0)\} \tag{14}$$

$$= b\{\xi - \xi^2\}V + b\xi^2 V_0. \tag{15}$$

The next iteration for the boundary potential is, following (13),

$$V_2 = A_1/b. \tag{16}$$

Finally we get the general form

$$V_n = \{\xi - \xi^2 + \xi^3 - \cdots - (-\xi)^{n+1}\}V + \xi^{n+1}V_0 \tag{17}$$

which in the limit as $n \to \infty$ becomes

$$V_\infty = \frac{\xi}{1 + \xi}V = (a/b)V \tag{18}$$

as long as $\xi < 1$ (i.e., $b > 2a$). This is the true value (cf. (5)).

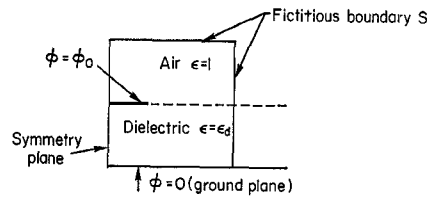

Fig. 3. Microstrip configuration.

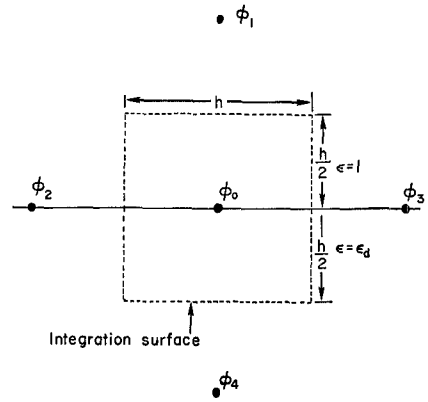

Fig. 4. Surface used to derive point difference equation on dielectric interface.

This example illustrates how the procedure works and indicates that it does not always converge. In our experience we have observed no convergence failures in computer solutions of real problems since we have kept the fictitious boundary a moderate distance from the real charges. However, we have found that the speed of convergence can be improved, as suggested by the sequence of values V_n in (17), by underrelaxing the corrections to the boundary potential.

For problems in which it is convenient or necessary to choose a fictitious boundary that excludes some of the charged regions but in which it is possible to estimate the charges omitted, the contribution of these charges to the potential at the fictitious boundary is added.

As an example of the complete computer solution procedure, consider the electrostatic problem for the microstrip shown in Fig. 3. Within either the dielectric or the air, the potential satisfies Laplace's equation; hence (1) and (3) are valid. Along the air–dielectric interface, the differential equation is

$$\nabla \cdot (\epsilon \nabla \phi) = 0. \tag{19}$$

In order to find the corresponding difference equation, we integrate (19) over a square box the same size as the grid boxes, but centered on the point of interest on the interface as shown in Fig. 4. Applying Gauss' theorem in two dimensions then gives

$$\int \nabla \cdot (\epsilon \nabla \phi)\, dS = \int (\epsilon \nabla \phi) \cdot \hat{n}\, dl = 0 \tag{20}$$

where $\hat{n}$ is the unit outward normal on the perimeter l of the box. If ϕ is replaced on each side of the box by its two-point difference value obtained at the center of the side, the second integral becomes

$$\left(\frac{\phi_0 - \phi_1}{h}\right)h + \left(\frac{\phi_0 - \phi_2}{h}\right)\cdot\frac{h}{2} + \left(\frac{\phi_0 - \phi_2}{h}\right)\frac{\epsilon_d h}{2}$$

$$+ \left(\frac{\phi_0 - \phi_4}{h}\right)\epsilon_d h + \left(\frac{\phi_0 - \phi_2}{h}\right)\frac{h}{2} + \left(\frac{\phi_0 - \phi_3}{h}\right)\frac{\epsilon_d h}{2} = 0 \tag{21}$$

or

$$4\phi_0 = \frac{\phi_1 + \epsilon_d\phi_4}{(\epsilon_d + 1)/2} + \phi_2 + \phi_3 \tag{22}$$

where h is the distance between grid points. The set of equations from each point on the grid is solved by the SOR technique subject to

the boundary constraints

$\phi = \phi_0$ on top conductor

$\phi = 0$ on ground plane

$\frac{\partial \phi}{\partial n} = 0$ on symmetry plane

$\phi = \phi(s)$ for points s on the boundary S.

The only charges in the region are on the top conductor, the ground plane, and the air–dielectric interface. The effects of the charges on the ground plane are represented by images of the charges on the top conductor and on the air–dielectric interface. The charges are given by

$$\rho = \nabla^2 \phi \tag{23}$$

or

$$\rho_0 = (\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0)/h^2. \tag{24}$$

The potential at a point on S due to a charge ρ_0 and its image is given by

$$\phi(s) = \frac{\rho_0}{2\pi} \ln \frac{R}{R'} \tag{25}$$

where R is the distance from the charge ρ_0 to s and R' is the distance from its image to s. The potential goes as ln (R) rather than $1/R$ since we are dealing, equivalently, with either a two-dimensional problem or a three-dimensional problem in which ρ is a line charge density extending to infinity normal to the page. The total potential on each point of S is found by summing (25) for each point on the conductor and on the interface and for their symmetrical points to the left of the symmetry plane. Since the charge on the interface extends outside the surface S, it could not be computed directly from (25) everywhere. However, far from the center conductor (and as will be seen, this does not mean very far), the charges on the interface go as $1/r^2$ where r is the distance from the charge to the center conductor. With this known dependence, the charges outside the boundary can be calculated from those on the grid point just inside the boundary. Far enough from the boundary, they can be neglected entirely.

In Fig. 5 we show the results of such a calculation employing the self-consistent boundary-value technique reported in this correspondence. The grid used is quite coarse for convenience in displaying the results. For accurate work a finer mesh is necessary. The potential ϕ_0 of the top conductor is defined to be 900. The top conductor has a half-width of three and is three grid points above the ground plane. The dielectric has $\epsilon_d = 16$. The charges outside the boundary are calculated as described above up to an additional distance equal to the number of points in the grid in the horizontal direction, effectively doubling the width of the grid as far as horizontal truncation of the charge is concerned.

The results are shown for four different choices of boundary. As is to be expected, the results for the interior are independent of the location of the boundary except for the smallest case (8×8 points) shown. The small deviations in that case result from inaccuracies in calculating the charge on the part of the interface outside the boundary. One can also see that the potential on the boundary is still substantial even with the most distant boundary. It would have been necessary to use a much bigger region to justify setting $\phi = 0$ on the outer boundary.

Although this correspondence has been written in terms of an electrostatic problem, the technique is also applicable to potential functions derivable from more general sources.

F. Sandy
J. Sage
Raytheon Research Div.
Waltham, Mass. 02154

488	480	457	419	372	320	270	226
563	554	525	475	406	335	271	218
657	647	614	550	444	342	261	199
771	763	736	665	477	330	231	163
900	900	900	900	468	271	169	120
591	587	568	513	343	217	137	86
292	288	274	240	175	117	75	48
0	0	0	0	0	0	0	0

(a)

484	476	452	415	366	314	263	218	179	148	123	102
560	550	521	471	402	329	264	210	167	133	107	87
655	645	612	546	440	337	254	191	145	111	86	67
770	762	734	663	474	325	224	156	111	80	59	42
900	900	900	900	466	267	160	99	62	40	26	21
591	587	568	512	342	214	132	82	51	32	20	11
292	288	274	240	174	115	73	46	29	18	11	6
0	0	0	0	0	0	0	0	0	0	0	0

(b)

300	298	290	277	261	242	222	201	181	163	145	130
333	330	319	303	282	258	233	209	185	163	144	127
373	369	355	334	307	276	245	214	187	162	140	122
422	416	399	370	334	294	255	218	185	157	133	114
483	476	453	415	366	313	262	217	179	148	123	102
560	550	521	471	401	329	264	210	167	133	107	87
655	645	612	546	439	337	254	191	145	111	86	67
769	762	734	663	474	325	224	156	111	80	59	43
900	900	900	900	466	267	160	99	62	40	26	21
591	587	568	512	342	214	132	82	51	32	20	11
292	288	274	240	174	115	73	46	29	18	11	6
0	0	0	0	0	0	0	0	0	0	0	0

(c)

301	298	290	277	261	242	222	202	182	163	146	130	116	104	93	84	75	68
333	330	320	303	283	259	234	209	185	164	144	127	112	99	88	78	70	63
373	369	355	334	307	276	245	215	187	162	141	122	107	93	82	72	64	57
422	416	399	371	335	295	255	218	186	156	134	115	98	85	74	64	56	50
483	476	453	415	366	313	262	218	180	149	124	104	87	74	64	55	48	42
560	550	521	471	401	329	264	210	167	134	108	89	73	61	52	44	38	33
655	645	612	546	440	337	254	191	145	112	87	69	56	46	38	32	27	23
769	762	734	663	474	325	224	156	111	80	60	45	35	28	22	18	15	12
900	900	900	900	466	267	160	99	62	39	25	17	11	8	6	4	3	4
591	587	568	512	342	214	132	82	51	32	21	13	9	6	4	3	2	1
292	288	274	240	174	115	73	46	29	18	11	7	5	3	2	1	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(d)

Fig. 5. Potentials in microstrip calculated using several fictitious boundaries. (a) 8 × 8 grid. (b) 12 × 8 grid. (c) 12 × 12 grid. (d) 18 × 12 grid.

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New Design Equations for a Class of Microwave Filters

***Abstract*—New approximate design equations for a class of microwave bandpass filters are presented. The filters are 1) dual forms of half-wave parallel-coupled resonator filters, 2) one form of interdigital filter, and 3) dual forms of direct-coupled stub filters. The advantages derived from using the new equations are 1) exact realization of the specified design bandwidth and 2) improved passband voltage standing-wave ratio (VSWR) response in the vicinity of band edge. Experimental data are presented for a trial filter design having 7 resonators, 40-percent bandwidth, and passband VSWR of 1.2.**

I. Introduction

The design equations presented in this correspondence apply to the following microwave filters: 1) dual forms of half-wave parallel-coupled resonator filters (Fig. 1), 2) the interdigital filter (Fig. 2), and 3) dual forms of direct-coupled stub filters (Fig. 3). All of these

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